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Nested semantics over finite trees are equationally hard

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Abstract

This paper studies nested simulation and nested trace semantics over the language BCCSP, a basic formalism to express finite process behaviour. It is shown that none of these semantics affords finite (in)equational axiomatizations over BCCSP. In particular, for each of the nested semantics studied in this paper, the collection of sound, closed (in)equations over a singleton action set is not finitely based.

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1. Introduction

Labelled transition systems (LTSs) [23] are a fundamental model of concurrent computation, which is widely used in light of its flexibility and applicability. In particular, they are the prime model

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underlying Plotkin's Structural Operational Semantics [30] and, following Milner's pioneering work on CCS [25], are by now the standard semantic model for various process description languages.

LTSs model processes by explicitly describing their states and their transitions from state to state, together with the actions that produced them. Since this view of process behaviours is very detailed, several notions of behavioural equivalence and preorder have been proposed for LTSs. The aim of such behavioural semantics is to identify those (states of) LTSs that afford the same "observations", in some appropriate technical sense. The lack of consensus on what constitutes an appropriate notion of observable behaviour for reactive systems has led to a large number of proposals for behavioural equivalences for concurrent processes. (See the study [14], where van Glabbeek presents the linear time-branching time spectrum—a lattice of known behavioural equivalences and preorders over LTSs, ordered by inclusion.)

One of the criteria that has been put forward for studying the mathematical tractability of the behavioural equivalences in the linear time-branching time spectrum is that they afford elegant, finite equational axiomatizations over fragments of process algebraic languages. Equationally based proof systems play an important role in both the practice and the theory of process algebras. From the point of view of practice, these proof systems can be used to perform system verifications in a purely syntactic way, and form the basis of axiomatic verification tools like, e.g., PAM [24]. From the theoretical point of view, complete axiomatizations of behavioural equivalences capture the essence of different notions of semantics for processes in terms of a basic collection of identities, and this often allows one to compare semantics which may have been defined in very different styles and frameworks. A review of existing complete equational axiomatizations for many of the behavioural semantics in van Glabbeek's spectrum is offered in [14]. The equational axiomatizations offered ibidem are over the language BCCSP, a common fragment of Milner's CCS [25] and Hoare's CSP [20] suitable for describing finite synchronization trees, and characterize the differences between behavioural semantics in terms of a few revealing axioms.

The main omissions in this menagerie of equational axiomatizations for the behavioural semantics in van Glabbeek's spectrum are axiomatizations for 2-nested simulation semantics and possible futures semantics. The relation of 2-nested simulation was introduced by Groote and Vaandrager [17] as the coarsest equivalence included in completed trace equivalence for which the tyft/tyxt format is a congruence format. It thus characterizes the distinctions amongst processes that can be made by observing their termination behaviour in program contexts that can be built using a wide array of operators. (The interested reader is referred to [17] for motivation and the basic theory of 2-nested simulation.) 2-nested simulation can be decided over finite LTSs in time that is quadratic in their number of transitions [34], and can be characterized by a single parameterized modal logic formula [26]. However, no equational axiomatization for it has ever been proposed, even for the language BCCSP. Possible futures semantics, on the other hand, was proposed by Rounds and Brookes in [32] as far back as 1981, and it affords an elegant modal characterization in terms of a subset of Hennessy-Milner logic—in fact, since possible futures equivalence (respectively, preorder) coincides with the 2-nested trace equivalence (resp. the 2-nested trace preorder), the modal characterization of possible futures equivalence is a consequence of a more general, classic result due to Hennessy and Milner (see [18, Theorem 2.2, and page 148]) that will find application in the technical developments of this paper. As shown by Kannellakis and Smolka in [22], the problem of deciding possible futures equivalence and all of the other n-nested trace equivalences ($n \geqslant 1$) from [18] over finite state processes is PSPACE-complete.

In this paper, we offer, amongst other results, a mathematical justification for the lack of an equational axiomatization for the 2-nested simulation and possible futures equivalence and preorder even for the language of finite synchronization trees. More precisely, we show that none of these behavioural relations admits a finite (in)equational axiomatization over the language BCCSP. These negative results hold in a very strong form. Indeed, we prove that no finite collection of inequations that are sound with respect to the 2-nested simulation preorder can prove all of the inequalities of the form

$$a^{2m} \sqsubseteq a^{2m} + a^m \qquad (m \geqslant 0),$$

which are sound with respect to the 2-nested simulation preorder. Similarly, we establish a result to the effect that no finite collection of (in)equations that are sound with respect to the possible futures preorder or equivalence can be used to derive all of the sound inequalities of the form

$$a(a^m + a^{2m}) + aa^{3m} \sqsubseteq aa^{2m} + a(a^m + a^{3m}) \qquad (m \geqslant 0).$$

We then generalize these negative results to show that none of the n-nested simulation or trace preorders and equivalences from [17,18] (for $n \geqslant 2$) afford finite equational axiomatizations over the language BCCSP.

The import of these results is not only that the equational theory of the n-nested simulation and trace semantics is not finitely equationally axiomatizable, for $n \geqslant 2$, but neither is the collection of (in)equivalences that hold between BCCSP terms over one action and without occurrences of variables. This state of affairs should be contrasted with the elegant equational axiomatizations over BCCSP for most of the other behavioural equivalences in the linear time-branching time spectrum that are reviewed by van Glabbeek in [14]. Only in the case of additional, more complex operators, such as iteration or parallel composition, or in the presence of infinite sets of actions, are these equivalences known to lack a finite equational axiomatization; see, e.g., [3,8,11,13,31,33]. Of special relevance for concurrency theory are Moller's results to the effect that the process algebras CCS and ACP without the auxiliary left merge operator from [6] do not have a finite equational axiomatization modulo bisimulation equivalence [27,28]. Fokkink and Luttik have shown in [12] that the process algebra PA [7], which contains a parallel composition operator based on pure interleaving without communication and the left merge operator, affords an ω-complete axiomatization that is finite if so is the underlying set of actions. Aceto, Ésik and Ingólfsdóttir [2] proved that there is no finite equational axiomatization that is ω-complete for the max-plus algebra of the natural numbers, a result whose process algebraic implications are discussed in [1].

As shown in [17,18], the intersection of all of the n-nested simulation or trace equivalences or preorders over image-finite labelled transition systems, and therefore over the language BCCSP, is bisimulation equivalence. Hennessy and Milner proved in [18] that bisimulation equivalence is axiomatized over the language BCCSP by the four equations in Table 2. Thus, in light of the aforementioned negative results, this fundamental behavioural equivalence, albeit finitely based over BCCSP, is the intersection of sequences of relations that do not afford finite equational axiomatizations themselves. This observation begs the question of whether bisimulation equivalence over BCCSP is the limit of some sequence of finitely based behavioural equivalences that have been presented in the literature. In [18] Hennessy and Milner introduced an alternative sequence of relations that approximate bisimulation equivalence. These relations are based on a "bisimulation-like" matching of the *single steps* that processes may perform, whereas the n-nested trace equivalences require matchings of arbitrarily long *sequences of steps*. We prove in this study that, unlike the n-nested trace equivalences, these single-step based approximations of bisimulation equivalence are all finitely axiomatizable over the language BCCSP, provided that the set of actions is finite.

The paper is organized as follows. We begin by presenting preliminaries on the language BCCSP, (in)equational logic, and the notions of behavioural equivalence and preorder studied in this paper (Section 2). Our main results on the non-existence of finite (in)equational axiomatizations for the n-nested simulation and trace equivalence and preorder (for $n \geqslant 2$) are the topic of Sections 3–5. In Section 3 we prove that the 2-nested simulation preorder has no finite inequational axiomatization over the language BCCSP. Section 4 presents a non-finite axiomatizability result for the possible futures preorder and equivalence. We then offer a general result to the effect that all of the other n-nested semantics considered in this study have no finite (in)equational axiomatization (Section 5). The paper concludes with our proof of finite axiomatizability for the alternative approximations of bisimulation equivalence introduced by Hennessy and Milner in [18] (Section 6).

The work reported in this paper extends and improves upon the results presented in [4], where it was shown that 2-nested simulation semantics and the 3-nested simulation preorder are not finitely based over the language BCCSP. The aforementioned paper also offered conditional axiomatizations for the nested simulation semantics. Since we have been unable to obtain similar results for the nested trace semantics, we have decided to omit those conditional axiomatizations from this presentation.

2. Preliminaries

We begin by introducing the basic definitions and results on which the technical developments to follow are based.

2.1. The language BCCSP

The process algebra BCCSP is a basic formalism to express finite process behaviour. Its syntax consists of (process) terms that are constructed from a countably infinite set of (process) variables (with typical elements x, y, z), a constant $\mathbf{0}$, a binary operator $+$ called *alternative composition*, and unary *prefixing* operators a, where a ranges over some non-empty set A of *atomic actions*. We shall use the meta-variables t, u, v to range over process terms, and write $var(t)$ for the collection of variables occurring in the term t.

A process term is *closed* if it does not contain any variables. Closed terms will be typically denoted by p, q, r. Intuitively, closed terms represent completely specified finite process behaviours, where $\mathbf{0}$ does not exhibit any behaviour, $p + q$ combines the behaviours of p and q by offering an initial choice as to whether to behave like either of these two terms, and ap can execute action a to transform into p. This intuition for the operators of BCCSP is captured, in the style of Plotkin [30], by the transition rules in Table 1. These transition rules give rise to transitions between process terms. The operational semantics for BCCSP is thus given by the labelled transition system [23] whose states are terms, and whose A-labelled transitions are those that are provable using the rules

Table 1
Transition rules for BCCSP

$\dfrac{x \xrightarrow{a} x'}{x + y \xrightarrow{a} x'}$	$\dfrac{y \xrightarrow{a} y'}{x + y \xrightarrow{a} y'}$	$ax \xrightarrow{a} x$

in Table 1. Based on this labelled transition system, we shall consider BCCSP terms modulo a range of behavioural equivalences that will be introduced in Section 2.4.

A (closed) substitution is a mapping from process variables to (closed) BCCSP terms. For every term t and (closed) substitution σ, the (closed) term obtained by replacing every occurrence of a variable x in t with the (closed) term $\sigma(x)$ will be written $\sigma(t)$.

In the remainder of this paper, we let a^0 denote $\mathbf{0}$, and a^{m+1} denote $a(a^m)$. Following standard practice in the literature on CCS and related languages, trailing $\mathbf{0}$'s will often be omitted from terms. A *term over action a* is a BCCSP term that may only contain occurrences of the prefixing operator a. (We shall restrict our attention to these terms in the technical developments presented in Section 5.) For example, the term a^m is over action a, for each $m \geqslant 0$.

2.2. Inequational logic

An *axiom system* is a collection of inequations $t \sqsubseteq u$ over the language BCCSP. An inequation $p \sqsubseteq q$ is derivable from E, notation $E \vdash p \sqsubseteq q$, if it can be proven from the axioms in E using the rules of inequational logic (viz. reflexivity, transitivity, substitution, and closure under BCCSP contexts):

$$t \sqsubseteq t \qquad \frac{t \sqsubseteq u \quad u \sqsubseteq v}{t \sqsubseteq v} \qquad \frac{t \sqsubseteq u}{\sigma(t) \sqsubseteq \sigma(u)} \qquad \frac{t \sqsubseteq u}{at \sqsubseteq au}(a \in A)$$

$$\frac{t \sqsubseteq u}{t + r \sqsubseteq u + r} \qquad \frac{t \sqsubseteq u}{r + t \sqsubseteq r + u}.$$

Without loss of generality one may assume that substitutions happen first in inequational proofs, i.e., that the third rule may only be used when $(t \sqsubseteq u) \in E$. In this case $\sigma(t) \sqsubseteq \sigma(u)$ is called a *substitution instance* of an axiom in E.

Equational logic is like inequational logic, but with the extra rule of symmetry:

$$\frac{t \sqsubseteq u}{u \sqsubseteq t}.$$

In equational logic, the formula $t \sqsubseteq u$ is normally written $t \approx u$. Without loss of generality, one may assume that applications of symmetry happen first in equational proofs. Therefore, we can see equational logic as a special case of inequational logic, namely by postulating that for each axiom in E also its symmetric counterpart is present in E. In the remainder of this paper, we shall always tacitly assume this property of equational axiom systems.

An example of an (equational) axiom system over the language BCCSP is given in Table 2. As shown by Hennessy and Milner in [18], that axiom system is sound and complete for bisimulation equivalence over the language BCCSP.

Table 2
Axioms for BCCSP

A1	$x + y \approx y + x$
A2	$(x + y) + z \approx x + (y + z)$
A3	$x + x \approx x$
A4	$x + \mathbf{0} \approx x$

In the remainder of this paper, process terms are considered modulo associativity and commutativity of +, and modulo absorption of **0** summands. In other words, we do not distinguish $t + u$ and $u + t$, nor $(t + u) + v$ and $t + (u + v)$, nor $t + \mathbf{0}$ and t. This is justified because all of the behavioural equivalences we consider satisfy axioms A1, A2 and A4 in Table 2. In what follows, the symbol $=$ will denote syntactic equality modulo axioms A1, A2 and A4. We use a *summation* $\sum_{i\in\{1,\ldots,k\}} t_i$ to denote $t_1 + \cdots + t_k$, where the empty sum represents **0**. It is easy to see that, modulo the equations A1, A2 and A4, every BCCSP term t has the form $\sum_{i\in I} x_i + \sum_{j\in J} a_j t_j$, for some finite index sets I, J, terms $a_j t_j$ ($j \in J$) and variables x_i ($i \in I$). The terms $a_j t_j$ ($j \in J$) and variables x_i ($i \in I$) will be referred to as the *summands* of t.

It is well-known (cf., e.g., Section 2 in [15]) that if an (in)equation relating two closed terms can be proven from an axiom system E, then there is a closed proof for it.

In the proofs of some of our main results, it will be convenient to use a different formulation of the notion of provability of an (in)equation from a set of axioms. This we now proceed to define for the sake of clarity.

A *context* $C[]$ is a closed BCCSP term with exactly one occurrence of a hole $[]$ in it. For every context $C[]$ and closed term p, we write $C[p]$ for the closed term that results by placing p in the hole in $C[]$. It is not hard to see that an inequation $p \sqsubseteq q$ is provable from an inequational axiom system E iff there is a sequence $p_1 \sqsubseteq \cdots \sqsubseteq p_k$ ($k \geqslant 1$) such that

- $p = p_1$,
- $q = p_k$ and
- $p_i = C[\sigma(t)] \sqsubseteq C[\sigma(u)] = p_{i+1}$ for some closed substitution σ, context $C[]$ and pair of terms t, u with $t \sqsubseteq u$ an axiom in E ($1 \leq i < k$).

In what follows, we shall refer to sequences of the form $p_1 \sqsubseteq \cdots \sqsubseteq p_k$ as *inequational derivations.*

For later use, note that, using axioms A1, A2 and A4 in Table 2, every context can be proven equal either to one of the form $C[b([] + p)]$ or to one of the form $[] + p$, for some action b and closed BCCSP term p.

2.3. Traces of BCCSP terms

The transition relations $\xrightarrow{a}$ ($a \in A$) naturally compose to determine the possible effects that performing a sequence of actions may have on a BCCSP term.

Definition 1. For a sequence $s = a_1 \cdots a_k \in A^*$ ($k \geqslant 0$), and BCCSP terms t, t', we write $t \xrightarrow{s} t'$ iff there exists a sequence of transitions

$$t = t_0 \xrightarrow{a_1} t_1 \xrightarrow{a_2} \cdots \xrightarrow{a_k} t_k = t'.$$

If $t \stackrel{s}{\longrightarrow} t'$ holds for some BCCSP term t', then s is a *trace* of t. We write $traces(t)$ for the set of traces of a term t.

The following lemma, whose proof is standard, relates the transitions of a term of the form $\sigma(t)$ to those of t and those of the terms $\sigma(x)$, with x a variable occurring in t.

Lemma 2. *For every BCCSP term t, substitution σ, and sequence of actions s, the following statements hold:*

(1) *if $t \stackrel{s}{\longrightarrow} u$ for some term u, then $\sigma(t) \stackrel{s}{\longrightarrow} \sigma(u)$;*
(2) *if $\sigma(t) \stackrel{s}{\longrightarrow} u$ for some term u, then*
    (a) *either $t \stackrel{s}{\longrightarrow} t'$ for some t' with $u = \sigma(t')$,*
    (b) *or there are sequences of actions s_1, s_2 with s_2 non-empty and $s = s_1 s_2$, a term t' and a variable x such that $t \stackrel{s_1}{\longrightarrow} x + t'$ and $\sigma(x) \stackrel{s_2}{\longrightarrow} u$.*

2.4. Behavioural semantics

Labelled transition systems describe the operational behaviour of processes in great detail. In order to abstract from irrelevant information on the way processes compute, a wealth of notions of behavioural equivalence or approximation have been studied in the literature on process theory. A systematic investigation of these notions is presented in [14], where van Glabbeek presents the so-called linear time-branching time spectrum, a lattice of known behavioural equivalences over labelled transition systems ordered by inclusion. In this study, we shall investigate a fragment of the notions of equivalence and preorder from [14], together with the family of the nested trace equivalences and preorders (see Definition 8). These we now proceed to present.

Definition 3. A binary relation R between closed terms is a *simulation* iff $p\,R\,q$ together with $p \stackrel{a}{\longrightarrow} p'$ imply that there is a transition $q \stackrel{a}{\longrightarrow} q'$ with $p'\,R\,q'$.

Groote and Vaandrager introduced in [17] a hierarchy of n-nested simulation preorders and equivalences for $n \geqslant 2$. These are defined thus:

Definition 4. For $n \geqslant 0$, we define the relation $\underset{\rightarrow}{\subset}_n$ inductively over closed BCCSP terms thus:

- $p \underset{\rightarrow}{\subset}_0 q$ for all p, q,
- $p \underset{\rightarrow}{\subset}_{n+1} q$ iff $p\,R\,q$ for some simulation R with R^{-1} included in $\underset{\rightarrow}{\subset}_n$.

The kernel of $\underset{\rightarrow}{\subset}_n$ (i.e., the equivalence $\underset{\rightarrow}{\subset}_n \cap (\underset{\rightarrow}{\subset}_n)^{-1}$) is denoted by $\leftrightarrows_n$.

The relation $\underset{\rightarrow}{\subset}_1$ is the well-known *simulation preorder* [29]. The relations $\underset{\rightarrow}{\subset}_2$ and $\leftrightarrows_2$ are the 2-*nested simulation preorder* and the 2-*nested simulation equivalence*, respectively. Groote and Vaandrager have characterized 2-nested semantics as the largest congruence with respect to the tyft/tyxt format of transition rules which is included in completed trace semantics—see [17] for details.

In the remainder of this paper we shall sometimes use, instead of Definition 4, the following more descriptive, fixed-point characterization of the n-nested simulation preorder ($n \geqslant 1$).

Proposition 5. *Let p, q be closed BCCSP terms, and $n \geqslant 0$. Then $p \overset{\subset}{\rightarrow}_{n+1} q$ iff*
(1) *for all $p \xrightarrow{a} p'$ there is a $q \xrightarrow{a} q'$ with $p' \overset{\subset}{\rightarrow}_{n+1} q'$, and*
(2) *$q \overset{\subset}{\rightarrow}_n p$.*

Proof. We prove the two implications separately.

- ($\Rightarrow$) Assume that $p \overset{\subset}{\rightarrow}_{n+1} q$. By definition, $p\, R\, q$ with R a simulation and R^{-1} included in $\overset{\subset}{\rightarrow}_n$. So if $p \xrightarrow{a} p'$, then $q \xrightarrow{a} q'$ with $p'\, R\, q'$, which implies

  $$p' \overset{\subset}{\rightarrow}_{n+1} q'.$$

  Moreover, since R^{-1} is included in $\overset{\subset}{\rightarrow}_n$, it follows that $q \overset{\subset}{\rightarrow}_n p$.
- ($\Leftarrow$) We define $p\, R\, q$ iff
  (1) for all $p \xrightarrow{a} p'$ there is a $q \xrightarrow{a} q'$ with $p' \overset{\subset}{\rightarrow}_{n+1} q'$, and
  (2) $q \overset{\subset}{\rightarrow}_n p$.
  Suppose now that $p\, R\, q$. If $p \xrightarrow{a} p'$, then by the definition of R we have $q \xrightarrow{a} q'$ with $p' \overset{\subset}{\rightarrow}_{n+1} q'$. Since we have already proven the 'only if' implication, we may conclude that $p'\, R\, q'$. So R is a simulation. Furthermore, by (2) above R^{-1} is included in $\overset{\subset}{\rightarrow}_n$. Hence, we have that $p \overset{\subset}{\rightarrow}_{n+1} q$, which was to be shown. □

Example 6. Let $m \geqslant 1$. Define, for each $n \in \mathbb{N}$, the closed BCCSP terms p_n and q_n thus:

$$\begin{array}{llll} p_0 & = a^{2m-1}\mathbf{0} & q_0 & = a^{m-1}\mathbf{0} \\ p_{n+1} & = ap_n + aq_n & q_{n+1} & = ap_n. \end{array}$$

By induction on $n \in \mathbb{N}$ and using Proposition 5, it is not hard to check that $p_n \overset{\subset}{\rightarrow}_n q_n$, and thus that $q_n \overset{\subset}{\rightarrow}_{n+1} p_n$.

The terms p_n and q_n ($n \in \mathbb{N}$) defined above will play a crucial role in the proof of Theorem 38 to follow.

Possible futures semantics was introduced by Rounds and Brookes in [32], and is defined thus:

Definition 7. Let p be a closed BCCSP term. A *possible future* of p is a pair (s, X), where s is a sequence of actions and $X \subseteq A^*$, such that $p \xrightarrow{s} p'$ and $X = \mathit{traces}(p')$, for some p'.

Two closed terms p and q are related by the *possible futures preorder* (respectively, *possible futures equivalence*), written $p \preceq_{PF} q$ (resp., $p =_{PF} q$), if each possible future of p is also a possible future of q (resp., if p and q have the same possible futures).

The last notions of semantics we shall consider in this paper are the families of the n-nested trace equivalences and preorders. The n-nested trace equivalences were introduced by Hennessy and Milner in [18, p. 147] as a a tool to define bisimulation equivalence [25,29].

Definition 8. For every $n \geqslant 0$, the relations of n-nested trace equivalence, denoted by $=_n^T$, and n-nested trace preorder, denoted by $\preceq_n^T$, are defined inductively over closed BCCSP terms thus:

- $p =_0^T q$ and $p \preceq_0^T q$ for every p, q;
- $p =_{n+1}^T q$ iff for every sequence of actions $s \in A^*$:
  - if $p \xrightarrow{s} p'$ then there is a q' such that $q \xrightarrow{s} q'$ and $p' =_n^T q'$, and
  - if $q \xrightarrow{s} q'$ then there is a p' such that $p \xrightarrow{s} p'$ and $p' =_n^T q'$;
- $p \preceq_{n+1}^T q$ iff for every sequence of actions $s \in A^*$:
  - if $p \xrightarrow{s} p'$ then there is a q' such that $q \xrightarrow{s} q'$ and $p' =_n^T q'$.

Note that the relations $=_1^T$ and $=_2^T$ are just trace equivalence (the equivalence that equates two terms having the same traces—see [14,19]) and possible futures equivalence, respectively, whereas $\preceq_2^T$ is the possible futures preorder. Moreover, it is easy to see that, for every $n \geqslant 0$, the equivalence relation $=_n^T$ is the kernel of the preorder $\preceq_n^T$.

The following result is well-known—see, e.g., the references [17,18].

Proposition 9. *For every $n \geqslant 0$, the relations $\overset{\subseteq}{\to}_n$, $\leftrightarrows_n$, $=_n^T$, and $\preceq_n^T$ are preserved by the operators of BCCSP.*

The relations we have previously defined over closed BCCSP terms are extended to arbitrary BCCSP terms thus:

Definition 10. Let t, u be BCCSP terms, and let $\preceq$ be any of $\overset{\subseteq}{\to}_n$, $\leftrightarrows_n$, $=_n^T$, and $\preceq_n^T$ $(n \geqslant 0)$. The inequation $t \sqsubseteq u$ is *sound* with respect to $\preceq$, written $t \preceq u$, iff $\sigma(t) \preceq \sigma(u)$ for every closed substitution σ.

For instance, the inequation $x \sqsubseteq y$ is sound with respect to all of the 0-nested semantics defined above. Examples of (in)equations that are sound with respect to $\overset{\subseteq}{\to}_2$ are those in Table 2 and

$$a(x + y) \sqsubseteq a(x + y) + ax.$$

The following result collects some basic properties of nested simulation and nested trace semantics that will be useful in the technical developments to follow.

Proposition 11. *For all BCCSP terms t, u, and $n \geqslant 0$, the following statements hold*:

(1) *if $t \overset{\subseteq}{\to}_{n+1} u$, then $t \leftrightarrows_n u$;*
(2) *if $t \preceq_{n+1}^T u$, then $t =_n^T u$;*
(3) *if $t \overset{\subseteq}{\to}_n u$, then $t \preceq_n^T u$.*

Proof. Statement (1) is due to Groote and Vaandrager in [17], and statement (2) follows immediately from the definitions of the relations $\preceq_{n+1}^T$ and $=_n^T$. We therefore limit ourselves to presenting a proof of statement (3). To this end, observe, first of all, that in light of Definition 10, it is sufficient to prove the claim for closed BCCSP terms. Assume now that $p \overset{\subseteq}{\to}_n q$, where p, q are closed BCCSP terms. We prove $p \preceq_n^T q$ by induction on n. This is trivial if $n = 0$. Suppose therefore that $p \overset{\subseteq}{\to}_{n+1} q$.

Let s be a sequence of actions in A, and assume that $p \xrightarrow{s} p'$ for some p'. We aim at showing that $q \xrightarrow{s} q'$ for some q' with $p' =_n^T q'$.

Since $p \precsim_{n+1} q$ and $p \xrightarrow{s} p'$, using Proposition 5 and a simple induction on the length of s, we have that $q \xrightarrow{s} q'$ for some q' with $p' \precsim_{n+1} q'$. By statement (1) of the proposition, we may infer that $p' \leftrightarrows_n q'$. The inductive hypothesis now yields that $p' \preceq_n^T q' \preceq_n^T p'$. Since the relation $=_n^T$ is the kernel of $\preceq_n^T$, we may conclude that $p' =_n^T q'$, which was to be shown. □

2.5. A modal characterization of nested trace equivalence

In the proof of our main result in Section 5, we shall make use of the modal characterization of the n-nested trace equivalences proposed by Hennessy and Milner in [18, p. 148]. This we now introduce for the sake of completeness.

Definition 12. The set $\mathcal{L}$ of Hennessy-Milner formulae over alphabet A is defined by the following grammar:

$$\varphi ::= \top \mid \varphi \wedge \varphi \mid \neg\varphi \mid \langle a\rangle\varphi \ (a \in A).$$

The satisfaction relation $\models$ is the binary relation relating closed BCCSP terms and Hennessy–Milner formulae defined by structural induction on formulae thus:
- $p \models \top$, for every closed BCCSP term p,
- $p \models \varphi_1 \wedge \varphi_2$ iff $p \models \varphi_1$ and $p \models \varphi_2$,
- $p \models \neg\varphi$ iff it is not the case that $p \models \varphi$, and
- $p \models \langle a\rangle\varphi$ iff $p \xrightarrow{a} p'$ for some p' such that $p' \models \varphi$.

As an immediate consequence of the characterization theorem for bisimulation equivalence over image-finite labelled transitions systems shown by Hennessy and Milner [18, Theorem 2.2], two closed BCCSP terms are bisimulation equivalent if, and only if, they satisfy the same formulae in $\mathcal{L}$. We now introduce a family of sub-languages of $\mathcal{L}$ that yield modal characterizations of the n-nested trace equivalences for every $n \geqslant 0$.

Definition 13. For every $n \geqslant 0$, we define the set $\mathcal{L}_n$ of n-nested Hennessy-Milner formulae inductively thus:
- $\mathcal{L}_0$ contains only the formulae $\top$ and $\neg\top$, and
- $\mathcal{L}_{n+1}$ is given by the following grammar

$$\varphi ::= \top \mid \varphi \wedge \varphi \mid \neg\varphi \mid \langle a_1\rangle \cdots \langle a_k\rangle\psi \ (k \geqslant 0, a_1 \cdots a_k \in A^* \text{ and } \psi \in \mathcal{L}_n).$$

The following result is due to Hennessy and Milner [18].

Theorem 14. *Let p, q be closed BCCSP terms, and let $n \geqslant 0$. Then $p =_n^T q$ iff p and q satisfy the same formulae in the language $\mathcal{L}_n$.*

Remark 15. Note that, for every $n \geqslant 0$ and closed terms p, q, if each formula in $\mathcal{L}_n$ satisfied by p is also satisfied by q, then p and q satisfy the same formulae in the language $\mathcal{L}_n$. Indeed, assume that each formula in $\mathcal{L}_n$ satisfied by p is also satisfied by q, and that q satisfies $\varphi \in \mathcal{L}_n$. Using the closure of $\mathcal{L}_n$ with respect to negation, we have that $q \not\models \neg\varphi$, and therefore that $p \not\models \neg\varphi$. It follows that p satisfies φ, which was to be shown.

Although tempting, it would therefore be incorrect to assume that, for every $n \geqslant 0$ and closed terms p, q, it holds that $p \preceq_n^T q$ iff each formula in $\mathcal{L}_n$ satisfied by p is also satisfied by q.

To obtain a modal characterization of the n-nested trace preorders, consider the sub-languages $\mathcal{M}_n$ of $\mathcal{L}_n$ defined inductively thus:

- $\mathcal{M}_0$ contains only the formulae $\top$ and $\neg\top$, and
- $\mathcal{M}_{n+1}$ is given by the following grammar

$$\varphi ::= \top \mid \varphi \wedge \varphi \mid \langle a_1 \rangle \cdots \langle a_k \rangle \psi \ (k \geqslant 0, a_1 \cdots a_k \in A^* \text{ and } \psi \in \mathcal{L}_n).$$

Following the lines of the proof of Theorem 2.2 in [18], the interested reader will have little trouble in establishing that

For every $n \geqslant 0$ and closed terms p, q, it holds that $p \preceq_n^T q$ iff each formula in $\mathcal{M}_n$ satisfied by p is also satisfied by q.

2.6. Lengths, norm, and depth of terms

We now present some results on the relationships between the lengths of the completed traces, the depth and the norm of BCCSP terms that are related by the notions of semantics considered in this paper. These will find important applications in the proofs of our main results, and shed light on the nature of the identifications made by the nested simulation and trace semantics.

Definition 16. A sequence $s \in A^*$ is a *completed trace* of a term t iff $t \xrightarrow{s} t'$ holds for some term t' without outgoing transitions. We write *lengths*(t) for the set of lengths of the completed traces of a BCCSP term t.

Note that *lengths*(t) is non-empty for each BCCSP term t. Moreover, the only closed BCCSP term that has a completed trace of length 0 is $\mathbf{0}$. (Recall that we consider terms modulo absorption of $\mathbf{0}$-summands.)

Definition 17. The *depth* and the *norm* of a BCCSP term t, denoted by *depth*(t) and *norm*(t), are the lengths of the longest and of the shortest completed trace of t, respectively.

The following lemma states the basic relations between the behavioural semantics studied in this paper and the lengths, depth, and norm of terms that will be needed in the technical developments to follow.

Lemma 18. *Let $\preceq$ be any of $\preceq_n^T$, $=_n^T$, $\leftrightarrows_n$, and $\overset{\subset}{\rightarrow}_n$, for $n \geqslant 2$. If $t \preceq u$, then*

(a) $lengths(t) \subseteq lengths(u)$,
(b) $depth(t) = depth(u)$,
(c) $norm(t) \geqslant norm(u)$ *and*
(d) $var(t) = var(u)$.

Proof. In light of Proposition 11, it is sufficient to prove that the claims hold for the possible futures preorder, viz. the relation $\preceq_2^T$.

We argue, first of all, that claims (a)–(c) hold when $t \preceq_2^T u$. To this end, note that, by substituting $\mathbf{0}$ for the variables in t and u, we obtain closed terms p and q with $lengths(t) = lengths(p)$ and $lengths(u) = lengths(q)$. So it suffices to prove claims (a)–(c) with p and q in place of t and u, respectively. By Definition 10, we have that $p \preceq_2^T q$.

Assume now that $n \in lengths(p)$. Then there are a sequence $s \in A^*$ of length n and a closed term p' with no outgoing transitions such that $p \xrightarrow{s} p'$. As $p \preceq_2^T q$, there is a closed term q' such that $q \xrightarrow{s} q'$ and $p' =_1^T q'$. Recall that $p' =_1^T q'$ if, and only if, p' and q' have the same traces. It therefore follows that q' has no outgoing transitions, and that $n \in lengths(q)$, which was to be shown.

Claim (c) follows immediately from (a). To see that claim (b) holds, observe that if $p \preceq_2^T q$ for closed BCCSP terms p and q, then, by Proposition 11(2), p and q have the same non-empty finite sets of traces, and thus the same longest traces.

To prove claim (d), let t, u be BCCSP terms such that $t \preceq_2^T u$. Assume, towards a contradiction, that there is a variable x that occurs in only one of t and u. We shall exhibit a closed substitution σ such that $depth(\sigma(t)) \neq depth(\sigma(u))$, contradicting statement (b) of the lemma.

To this end, observe, first of all, that without loss of generality, we may assume that x occurs in t, say. Let m be a positive integer larger than $depth(t)$. By claim (b) of the lemma, we have that $depth(t) = depth(u) < m$ also holds.

Consider now the closed substitution σ that maps x to a^m, and all the other variables to $\mathbf{0}$. Using structural induction, it is a simple matter to prove that

$$\begin{aligned} depth(\sigma(t)) &\geqslant m \quad \text{and} \\ depth(\sigma(u)) &= depth(u) < m. \end{aligned}$$

By statement (b) of the lemma, it follows that $\sigma(t) \preceq_2^T \sigma(u)$ does not hold, contradicting our assumption that $t \preceq_2^T u$. $\square$

Remark 19. Note that $lengths(t) = lengths(u)$ and $norm(t) = norm(u)$ both hold, if $t =_2^T u$.

The restriction that $n \geqslant 2$ is necessary in the statement of Lemma 18(a) and (c). In fact, $aa + a \leftrightarrows_1 aa$, but

$$\begin{aligned} &lengths(aa + a) = \{1, 2\} \not\subseteq \{2\} = lengths(aa) \quad \text{and} \\ &norm(aa + a) < norm(aa). \end{aligned}$$

Statements (b) and (d) in Lemma 18 also hold for $=_1^T$. In fact, it is not hard to see that, for every t, u, if $t \preceq_1^T u$ then $depth(t) \leqslant depth(u)$ and $var(t) \subseteq var(u)$.

3. Non-finite axiomatizability of the 2-nested simulation preorder

In this section we prove that the 2-nested simulation preorder is not finitely inequationally axiomatizable. The following lemma will play a key role in the proof of this statement.

Lemma 20. *If $p \overset{\subseteq}{\rightarrow}_2 a^{2m} + a^m$, then either $p \leftrightarrows_2 a^{2m}$ or $p \leftrightarrows_2 a^{2m} + a^m$.*

Proof. The case $m = 0$ is trivial; we therefore focus on the case $m > 0$. We note, first of all, that if $q \overset{\subset}{\rightarrow}_2 a^k$ for some $k \geqslant 0$, then, by Lemma 18(a), q has only the completed trace a^k; clearly, this implies $a^k \overset{\subset}{\rightarrow}_2 q$, and hence $a^k \leftrightarrows_2 q$.

Consider now a transition $p \xrightarrow{a} p'$. Since $p \overset{\subset}{\rightarrow}_2 a^{2m} + a^m$, either $p' \overset{\subset}{\rightarrow}_2 a^{2m-1}$ or $p' \overset{\subset}{\rightarrow}_2 a^{m-1}$. By Lemma 18(b), p has depth $2m$. So there is at least one transition $p \xrightarrow{a} p'$ with $p' \overset{\subset}{\rightarrow}_2 a^{2m-1}$.

If for all transitions $p \xrightarrow{a} p'$ we have $p' \overset{\subset}{\rightarrow}_2 a^{2m-1}$, then it follows that $p \overset{\subset}{\rightarrow}_2 a^{2m}$, and hence $p \leftrightarrows_2 a^{2m}$. On the other hand, if there exists a transition $p \xrightarrow{a} p''$ with $p'' \overset{\subset}{\rightarrow}_2 a^{m-1}$ (and so $a^{m-1} \overset{\subset}{\rightarrow}_2 p''$), then it follows that $a^{2m} + a^m \overset{\subset}{\rightarrow}_2 p$, and hence $p \leftrightarrows_2 a^{2m} + a^m$. $\square$

The idea behind our proof that the 2-nested simulation preorder is not finitely inequationally axiomatizable is as follows. Assume a finite inequational axiomatization E for BCCSP that is sound modulo $\overset{\subset}{\rightarrow}_2$. We show that, if m is sufficiently large, then, for all closed inequational derivations $a^{2m} \sqsubseteq p_1 \sqsubseteq \cdots \sqsubseteq p_k$ from E with $p_k \overset{\subset}{\rightarrow}_2 a^{2m} + a^m$, we have that $p_k \leftrightarrows_2 a^{2m}$. Since $a^{2m} + a^m \overset{\subset}{\not\rightarrow}_2 a^{2m}$, it follows that $a^{2m} \sqsubseteq a^{2m} + a^m$ cannot be derived from E. However, $a^{2m} \overset{\subset}{\rightarrow}_2 a^{2m} + a^m$.

The following lemma is the crux in the implementation of the aforementioned proof idea.

Lemma 21. *Let $t \sqsubseteq u$ be sound modulo $\overset{\subset}{\rightarrow}_2$. Let m be greater than the depth of t. Assume that $C[\sigma(u)] \overset{\subset}{\rightarrow}_2 a^{2m} + a^m$, for some closed substitution σ. Then $C[\sigma(t)] \leftrightarrows_2 a^{2m}$ implies $C[\sigma(u)] \leftrightarrows_2 a^{2m}$.*

Proof. Let $C[\sigma(t)] \leftrightarrows_2 a^{2m}$; we prove $C[\sigma(u)] \leftrightarrows_2 a^{2m}$. Since $C[\sigma(u)] \overset{\subset}{\rightarrow}_2 a^{2m} + a^m$, it is sufficient to show that $a^{2m} + a^m \overset{\subset}{\not\rightarrow}_2 C[\sigma(u)]$. In fact, if $C[\sigma(u)] \overset{\subset}{\rightarrow}_2 a^{2m} + a^m$ and $a^{2m} + a^m \overset{\subset}{\not\rightarrow}_2 C[\sigma(u)]$, by Lemma 20 it follows that $C[\sigma(u)] \leftrightarrows_2 a^{2m}$, which is to be shown. We prove $a^{2m} + a^m \overset{\subset}{\not\rightarrow}_2 C[\sigma(u)]$ by distinguishing two cases, depending on the form of the context $C[]$.

- *Case 1*: Suppose $C[]$ is of the form $C'[b([] + r)]$.

  In this case, we shall prove $a^{2m} + a^m \overset{\subset}{\not\rightarrow}_2 C[\sigma(u)]$ by arguing that $a^{m-1} \overset{\subset}{\not\rightarrow}_2 q'$ holds for each q' such that $C[\sigma(u)] \xrightarrow{a} q'$. To this end, consider a transition

  $$C[\sigma(u)] \xrightarrow{a} q'.$$

  Then $q' = D[\sigma(u)]$ for some context $D[]$, and, because of the form of the context $C[]$, we may infer that

  $$C[\sigma(t)] \xrightarrow{a} p' = D[\sigma(t)].$$

  As $\sigma(t) \overset{\subset}{\rightarrow}_2 \sigma(u)$ by the soundness of $t \sqsubseteq u$ with respect to $\overset{\subset}{\rightarrow}_2$, and $p' \overset{\subset}{\rightarrow}_2 q'$ by Proposition 9, Lemma 18(b) yields that p' and q' have the same depth. Since $C[\sigma(t)] \leftrightarrows_2 a^{2m}$, it follows by Proposition 5 that $p' \overset{\subset}{\rightarrow}_2 a^{2m-1}$. So by Lemma 18(b), we have that

  $$depth(p') = depth(q') = 2m - 1.$$

  As $depth(a^{m-1}) \neq 2m - 1$, another application of Lemma 18(b) yields that

  $$a^{m-1} \overset{\subset}{\not\rightarrow}_2 q'.$$

Since this holds for all transitions $C[\sigma(u)] \xrightarrow{a} q'$, and $a^{2m} + a^m \xrightarrow{a} a^{m-1}$, using Proposition 5 we may therefore conclude that $a^{2m} + a^m \not\overset{\subsetneq}{\to}_2 C[\sigma(u)]$.

- *Case 2*: Suppose $C[]$ is of the form $[] + r$.

  In this case, we shall prove $a^{2m} + a^m \not\overset{\subsetneq}{\to}_2 C[\sigma(u)]$ by arguing that the norm of $C[\sigma(u)]$ is larger than m.

  To this end, observe, first of all, that, as $t \overset{\subsetneq}{\to}_2 u$ by our assumptions, statements (b) and (d) in Lemma 18 imply that $depth(t) = depth(u)$, and moreover that t and u contain exactly the same variables. We proceed with the proof by distinguishing two cases, depending on whether $norm(\sigma(t)) = 0$ or not.

  - *Case norm* $(\sigma(t)) = 0$.

    In this case, t has the form $\sum_{i\in I} x_i$ for some finite index set I, and variables x_i $(i \in I)$ with $norm(\sigma(x_i)) = 0$ for each $i \in I$.

    Since $t \sqsubseteq u$ is sound with respect to $\overset{\subsetneq}{\to}_2$, statements (c)–(d) in Lemma 18 yield that $t = u$ modulo axiom A3. Since axiom A3 is sound with respect to $\leftrightarrows_2$, using Proposition 9 we may therefore conclude that

    $$a^{2m} + a^m \not\overset{\subsetneq}{\to}_2 a^{2m} \leftrightarrows_2 C[\sigma(t)] \leftrightarrows_2 C[\sigma(u)],$$

    which was to be shown.

  - *Case norm* $(\sigma(t)) > 0$.

    Since $\sigma(t) + r \leftrightarrows_2 a^{2m}$, Lemma 18(c) yields that $norm(\sigma(t)) \geqslant 2m$, and either $norm(r) \geqslant 2m$ or $norm(r) = 0$. By the soundness of $t \sqsubseteq u$ with respect to $\overset{\subsetneq}{\to}_2$, and the assumption that $norm(\sigma(t)) > 0$, it follows that $depth(\sigma(t)) = depth(\sigma(u)) > 0$. Hence $\sigma(u) \neq \mathbf{0}$, and therefore we have that $norm(\sigma(u)) > 0$. As $\sigma(u) + r \overset{\subsetneq}{\to}_2 a^{2m} + a^m$, again using Lemma 18(c), we infer that

    $$norm(\sigma(u)) \geqslant m.$$

    Since $depth(t) < m$ and $norm(\sigma(t)) \geqslant 2m$, for each variable $x \in var(t) = var(u)$ we have $norm(\sigma(x)) > m$.

    By the facts that $depth(u) = depth(t) < m$ and $norm(\sigma(u)) \geqslant m$, each completed trace of $\sigma(u)$ must become, after less than m transitions, a completed trace of a $\sigma(x)$ with $x \in var(u)$. Since for all $x \in var(u) = var(t)$ we have $norm(\sigma(x)) > m$, it follows that $norm(\sigma(u)) > m$. Since moreover $norm(r) \geqslant 2m$ or $norm(r) = 0$, we have $norm(\sigma(u) + r) > m$. As $a^{2m} + a^m$ has norm m, by Lemma 18(a) we may conclude that $a^{2m} + a^m \not\overset{\subsetneq}{\to}_2 \sigma(u) + r$, which was to be shown. $\square$

Remark 22. The inequation $ax \sqsubseteq ax + a^1$ is sound modulo $\overset{\subsetneq}{\to}_2$. However, $a^4 \not\leftrightarrows_2 a^4 + a^1$. So the proviso in the statement of Lemma 21 that $C[\sigma(u)] \overset{\subsetneq}{\to}_2 a^{2m} + a^m$ cannot be omitted. (Note that $a^4 + a^1 \not\overset{\subsetneq}{\to}_2 a^4 + a^2$.)

Theorem 23. *BCCSP modulo the* 2*-nested simulation preorder is not finitely inequationally axiomatizable.*

Proof. Let E be a finite inequational axiomatization for BCCSP that is sound modulo $\overset{\subsetneq}{\to}_2$. Let $m > \max\{depth(t) \mid t \sqsubseteq u \in E\}$.

By Lemma 21, and using induction on the length of derivations, it follows that if the closed inequation $a^{2m} \sqsubseteq r$ can be derived from E and $r \overset{\subsetneq}{\to}_2 a^{2m} + a^m$, then $r \leftrightarrows_2 a^{2m}$. As Lemma 18(c) yields that $a^{2m} + a^m \not\overset{\subsetneq}{\to}_2 a^{2m}$, it follows that $a^{2m} \sqsubseteq a^{2m} + a^m$ cannot be derived from E. Since $a^{2m} \overset{\subsetneq}{\to}_2 a^{2m} + a^m$, we may conclude that E is not complete modulo $\overset{\subsetneq}{\to}_2$. $\square$

4. Possible future semantics is not finitely based

Throughout this section, we let $\preceq$ be either the possible futures preorder, or possible futures equivalence. Our order of business in this section will be to prove that $\preceq$ has no finite (in)equational axiomatization over BCCSP. The idea behind the proof of this claim is as follows. Assume that E is a finite inequational axiomatization for BCCSP that is sound modulo $\preceq$. We show that, if m is sufficiently large, then, for all closed inequations $p \sqsubseteq q$ that can be derived from E the following invariant property holds:

> If $lengths(q) \subseteq \{m+1, 2m+1, 3m+1\}$, and there is a p' such that $p \xrightarrow{a} p'$, $norm(p') = m$, and $depth(p') \leqslant 2m$, then there is a q' such that $q \xrightarrow{a} q'$, $norm(q') = m$ and $depth(q') \leqslant 2m$.

However, we shall exhibit a pair of closed terms that are related by $\preceq$, and do not satisfy the above property. This will allow us to conclude that E is not complete with respect to $\preceq$.

The following lemma characterizes some properties of the inequations that are sound with respect to $\preceq$ that will be useful in the proof of the main result of this section (Theorem 25 to follow).

Lemma 24. *Let the axiom $t \sqsubseteq u$ be sound modulo $\preceq$. Let $t = \Sigma_{i\in I} x_i + \Sigma_{j\in J} a_j t_j$ and $u = \Sigma_{k\in K} y_k + \Sigma_{\ell\in L} b_\ell u_\ell$, and let x be a variable. Then*
(a) $\{x_i \mid i \in I\} \subseteq \{y_k \mid k \in K\}$, *and*
(b) *for each $j \in J$ with $x \in var(t_j)$ there is an $\ell \in L$ such that $a_j = b_\ell$, $x \in var(u_\ell)$ and $var(u_\ell) \subseteq var(t_j)$.*

Proof. Let $t \sqsubseteq u$ be sound modulo $\preceq$, and let x be a variable. We prove the two statements of the lemma separately.

- **Proof of Claim (a).** Assume, towards a contradiction, that the variable x is contained in $\{x_i \mid i \in I\}$, but not in $\{y_k \mid k \in K\}$. We shall exhibit a closed substitution σ such that $\sigma(t) \not\preceq \sigma(u)$, contradicting our assumption that $t \sqsubseteq u$ is sound modulo $\preceq$.

  To this end, pick a positive integer $m > depth(t)$. Since $t \sqsubseteq u$ is sound modulo $\preceq$, by Lemma 18(b) we have that $m > depth(u)$ also holds. Consider the closed substitution σ that maps x to a^m, and all the other variables to $\mathbf{0}$. Since $x = x_i$ for some $i \in I$, we have that $m \in lengths(\sigma(t))$. On the other hand, $m \notin lengths(\sigma(u))$ because, as x is not contained in $\{y_k \mid k \in K\}$, every completed trace of $\sigma(u)$ is either one of u itself (and is thus shorter than m) or has a^m has a proper suffix (and is thus longer than m). By Lemma 18(a), it follows that $\sigma(t) \preceq \sigma(u)$ does not hold, contradicting our assumption that $t \sqsubseteq u$ is sound modulo $\preceq$.

- **Proof of Claim (b).** Assume, towards a contradiction, that there is a $j \in J$ with $x \in var(t_j)$ such that, for each $\ell \in L$ with $a_j = b_\ell$ either $x \notin var(u_\ell)$ or $var(u_\ell) \not\subseteq var(t_j)$. We shall exhibit a closed substitution σ such that $\sigma(t) \not\preceq_2^T \sigma(u)$, contradicting our assumption that $t \sqsubseteq u$ is sound modulo $\preceq$.

Let m be a positive integer larger than $depth(t)$. Since $t \sqsubseteq u$ is sound modulo $\preceq$, by Lemma 18(b) we have that $m > depth(u)$ also holds. Consider the closed substitution mapping x to a^m, all of the variables not occurring in t_j to a^{2m}, and all the other variables to $\mathbf{0}$. Note that $\sigma(t) \xrightarrow{a_j} \sigma(t_j)$, by Lemma 2. Moreover, since $x \in var(t_j)$ and

$$depth(t_j) \leqslant depth(t) - 1 \leqslant m - 2,$$

it is easy to see that

$$m \leqslant depth(\sigma(t_j)) \leqslant 2m - 2. \tag{1}$$

We claim that if $\sigma(u) \xrightarrow{a_j} p$, then $depth(\sigma(t_j)) \neq depth(p)$. This shows that $\sigma(t) \not\precsim_2^T \sigma(u)$ because no p with $\sigma(u) \xrightarrow{a_j} p$ can have the same traces as $\sigma(t_j)$ (see Remark 19), contradicting our assumption that $t \sqsubseteq u$ is sound modulo $\preceq$.

To prove our claim, we consider the possible origins of a transition $\sigma(u) \xrightarrow{a_j} p$.

- *Case 1*: $\sigma(u) \xrightarrow{a_j} p$ because $\sigma(y_k) \xrightarrow{a_j} p$, for some $k \in K$. In this case, by the definition of σ, we have that $depth(p) \in \{m-1, 2m-1\}$. By (1), we may infer that $depth(\sigma(t_j)) \neq depth(p)$, as claimed.
- *Case 2*: $\sigma(u) \xrightarrow{a_j} p$ because $p = \sigma(u_\ell)$ for some $\ell \in L$ such that $a_j = b_\ell$ and either $x \notin var(u_\ell)$ or $var(u_\ell) \not\subseteq var(t_j)$. In this case, by the definition of σ and using that $depth(u) < m$, we have that $depth(p)$ is either smaller than $m-1$ (if $x \notin var(u_\ell)$ and $var(u_\ell) \subseteq var(t_j)$) or larger than $2m-1$ (if $var(u_\ell) \not\subseteq var(t_j)$). Again, by (1), we may infer that $depth(\sigma(t_j)) \neq depth(p)$, as claimed.

This completes the proof. □

We are now in a position to prove the promised result to the effect that possible futures semantics is not finitely based over the language BCCSP.

Theorem 25. *BCCSP modulo $\preceq$ is not finitely (in)equationally axiomatizable.*

Proof. Let E be a finite inequational axiomatization for BCCSP that is sound modulo $\preceq$. Let $m > \max\{depth(t), depth(u) \mid (t \sqsubseteq u) \in E\}$.

We have that

$$a(a^m + a^{2m}) + aa^{3m} \preceq aa^{2m} + a(a^m + a^{3m})$$

because both processes have the same possible futures. Nevertheless,

$$E \nvdash a(a^m + a^{2m}) + aa^{3m} \sqsubseteq aa^{2m} + a(a^m + a^{3m}).$$

This follows immediately from the following

Claim 26. *Assume that $E \vdash p \sqsubseteq q$, $lengths(q) \subseteq \{m+1, 2m+1, 3m+1\}$, and there is a p' such that $p \xrightarrow{a} p'$, $norm(p') = m$ and $depth(p') \leqslant 2m$. Then there is a q' such that $q \xrightarrow{a} q'$, $norm(q') = m$ and $depth(q') \leqslant 2m$.*

Proof of the claim. Using induction on the length of inequational derivations, the soundness of E with respect to $\preceq$ and Lemma 18(a), it suffices to consider the case that $p = C[\sigma(t)]$ and $q = C[\sigma(u)]$ for a BCCSP context $C[]$, a closed substitution σ, and an axiom $(t \sqsubseteq u) \in E$. We proceed by distinguishing two sub-cases, depending on the form of the context $C[]$.

- *Case 1:* Suppose $C[]$ is of the form $C'[b([] + r)]$.
  Let p' be as in the statement of the claim. Then $p' = D[\sigma(t)]$ for some context $D[]$, and, because of the form of the context $C[]$, we may infer that

  $$q = C[\sigma(u)] \xrightarrow{a} q' = D[\sigma(u)].$$

  By the soundness of E and the fact that $\preceq$ is preserved by the operators of BCCSP (Proposition 9), we have that $p' \preceq q'$. Therefore

  $$norm(q') \leqslant norm(p') = m \quad \text{and} \quad depth(q') = depth(p') \leqslant 2m$$

  both hold by statements (b) and (c) in Lemma 18. As $norm(q) \geqslant m + 1$ it follows that $norm(q') = m$, and we are done.
- *Case 2:* Suppose $C[]$ is of the form $[] + r$.
  Let $t = \Sigma_{i \in I} x_i + \Sigma_{j \in J} a_j t_j$ and $u = \Sigma_{k \in K} y_k + \Sigma_{\ell \in L} b_\ell u_\ell$. Consider a transition $\sigma(t) + r \xrightarrow{a} p'$ as in the statement of the claim. We distinguish three possible cases, depending on the origin of this transition.
  - *Case 2.1:* Assume that $r \xrightarrow{a} p'$. Then $q \xrightarrow{a} p'$ and we are done.
  - *Case 2.2:* Assume that $\sigma(x_i) \xrightarrow{a} p'$ for some $i \in I$. By Lemma 24(a) and the soundness of $t \sqsubseteq u$ with respect to $\preceq$, we have that $x_i = y_k$ for some $k \in K$. It follows that $q \xrightarrow{a} p'$, and we are done.
  - *Case 2.3:* Assume that $p' = \sigma(t_j)$ for some $j \in J$. As $norm(\sigma(t_j)) = m$ and

    $$depth(t_j) < depth(t) < m,$$

    there must be a variable $x \in var(t_j)$ such that $1 \leqslant norm(\sigma(x)) \leqslant m$. By statement (b) in Lemma 24, there is an $\ell \in L$ such that $a = b_\ell, x \in var(u_\ell)$ and $var(u_\ell) \subseteq var(t_j)$. Take $q' = \sigma(u_\ell)$. Then $q \xrightarrow{a} q'$. Since $x \in var(u_\ell)$, we have that

    $$norm(q') \leqslant depth(u_\ell) + norm(\sigma(x)) < 2m.$$

    Considering that

    $$lengths(q) \subseteq \{m + 1, 2m + 1, 3m + 1\},$$

    and thus $lengths(q') \subseteq \{m, 2m, 3m\}$, it must be the case that $norm(q') = m$.
    As $depth(\sigma(t_j)) \leqslant 2m$ by assumption, it follows that $depth(\sigma(y)) \leqslant 2m$ for each $y \in var(t_j)$. Since $var(u_\ell) \subseteq var(t_j)$, this also holds for each $y \in var(u_\ell)$. As $depth(u_\ell) < depth(u) < m$, this implies that $depth(\sigma(u_\ell)) < 3m$. Considering that $lengths(q') \subseteq \{m, 2m, 3m\}$, we may conclude that $depth(q') \leqslant 2m$.
    To sum up, we have proven that, also in this case, $q \xrightarrow{a} q'$, $norm(q') = m$ and $depth(q') \leqslant 2m$, which was to be shown. □

5. No nested semantics is finitely based

We now proceed to offer results to the effect that the language BCCSP modulo $=^T_n$ or $\leftrightarrows_n$, for $n \geqslant 2$, or $\preceq^T_n$ or $\overset{\subseteq}{\rightarrow}_n$, for $n \geqslant 3$, is not finitely (in)equationally axiomatizable. Rather than considering each of these behavioural relations in turn, we offer a general proof of non-finite axiomatizability that applies to all of them at once. The general strategy underlying such a proof is as follows. We prove that, for each $n \geqslant 2$, no finite collection of (in)equations that is sound with respect to $=^T_n$ (the coarsest relation amongst $=^T_n$, $\leftrightarrows_n$, $\preceq^T_{n+1}$, and $\overset{\subseteq}{\rightarrow}_{n+1}$) can prove all of the closed inequations of the form $p \sqsubseteq q$, with p and q BCCSP terms over action a, that are sound with respect to $\overset{\subseteq}{\rightarrow}_{n+1}$ (the finest relation amongst $=^T_n$, $\leftrightarrows_n$, $\preceq^T_{n+1}$, and $\overset{\subseteq}{\rightarrow}_{n+1}$). We remind the reader that $=^T_2$ is possible futures equivalence, so the main result of this section (Theorem 38) gives an alternative proof of non-finite axiomatizability for this behavioural equivalence over BCCSP.

In the proof of this result, we shall make use of the modal characterization of the relation $=^T_n$ given in Theorem 14. More specifically, we shall show that, for each $n \geqslant 2$ and finite axiom system E that is sound with respect to $=^T_n$, there is a formula ψ_n in the language $\mathcal{L}_{n+1}$ (see Definition 13) such that whenever E proves a closed inequation $p \sqsubseteq q$, with p and q BCCSP terms over action a, then, subject to some technical conditions on the lengths of the completed traces of q, it holds that p satisfies ψ_n if, and only if, so does q. We shall, however, show that this property does not hold for the inequation $q_n \overset{\subseteq}{\rightarrow}_{n+1} p_n$, where the terms p_n and q_n have been defined in Example 6. This will allow us to conclude that the sound inequation $q_n \sqsubseteq p_n$ cannot be derived from E, and thus that E is incomplete for $=^T_n$, $\leftrightarrows_n$, $\preceq^T_{n+1}$, and $\overset{\subseteq}{\rightarrow}_{n+1}$.

The technical implementation of the above idea will be based upon an induction on the length of the proof of closed inequations from the finite axiom system E. The crucial step in this proof will be to show that, subject to technical conditions, the aforementioned formula ψ_n is satisfied either by both terms in a substitution instance of an axiom in E or by neither of them. This case will be tackled by Lemma 37 to follow. We now introduce some technical notions, and preliminary results, that will be used in the proof of this crucial lemma.

Definition 27. We call a substitution σ *substantial* if $depth(\sigma(x)) > 0$ for all variables x.

For reasons of technical convenience, in the proofs of our non-finite axiomatizability results presented in this section we will only allow for the use of closed substantial substitutions in the rule of substitution. This does not limit the generality of those results because every finite inequational axiomatization E can be converted into a finite inequational axiomatization E' such that the closed substitution instances of the axioms of E are the same as the closed substantial substitution instances of the axioms of E' (when equating any closed subterm of depth 0 with $\mathbf{0}$). This is done by including in E' any inequation that can be obtained from an inequation in E by replacing all occurrences of any number of variables by $\mathbf{0}$.

Definition 28. Define the *depths* at which a subterm occurs in a BCCSP term as follows:
- t occurs in t at depth 0,
- if v occurs in t or u at depth d, then v occurs in $t + u$ at depth d,
- if v occurs in t at depth d then v occurs in at (with $a \in A$) at depth $d + 1$.

A BCCSP term t has a *unique depth allocation* if no variable occurs in t at two different depths.

For example, the term $ax + x$ does not have a unique depth allocation, as the variable x occurs both at depth 0 and at depth 1 in it, but $ax + y$ does.

The following lemma describes the interplay between the depths at which variables occur in a term t, and the lengths of terms of the form $\sigma(t)$, for some substantial substitution σ.

Lemma 29. *For every BCCSP term t and $d \geqslant 0$, the following statements hold:*
(1) *The term v occurs in t at depth d if, and only if, there are a term u and a sequence of actions s of length d such that $t \xrightarrow{s} v + u$.*
(2) *Let x be a variable, and let σ be a substitution. For every $n > 0$, if x occurs in t at depth d and $n \in lengths(\sigma(x))$ then $d + n \in lengths(\sigma(t))$.*

Proof. We prove the two statements separately. Recall that we consider equality of terms modulo axioms A1, A2 and A4 in Table 2.

- **Proof of statement 1.** We show the two implications separately.
  - ($\Rightarrow$) By induction on the definition of the depths at which v occurs in t.

    Assume that v occurs in t at depth d because $v = t$ and $d = 0$. Then, letting ε denote the empty string, we have that
    $$t \xrightarrow{\varepsilon} v = v + \mathbf{0},$$
    and we are done.
    Assume that v occurs in $t + t'$ at depth d because v occurs in t or t' at depth d. Suppose, without loss of generality, that v occurs in t at depth d. By induction, we have that there are a term u and a sequence of actions s of length d such that $t \xrightarrow{s} v + u$. If d is positive, we may immediately conclude that $t + t' \xrightarrow{s} v + u$. If $d = 0$, then $t = v + u$. It follows that $t + t' \xrightarrow{\varepsilon} v + u + t'$, and we are done.
    Assume that v occurs in at (with $a \in A$) at depth $d + 1$ because v occurs in t at depth d. By induction we have that there are a term u and a sequence of actions s of length d such that $t \xrightarrow{s} v + u$. It follows that $at \xrightarrow{as} v + u$, and we are done.
  - ($\Leftarrow$) Assume that there are a term u and a sequence of actions s of length d such that $t \xrightarrow{s} v + u$. We prove that v occurs in t at depth d by induction on d. Throughout the proof, we let $t = \sum_{i \in I} x_i + \sum_{j \in J} a_j t_j$.

    Base Case: $d = 0$. Since $t \xrightarrow{\varepsilon} v + u$, we have that
    $$t = \sum_{i \in I} x_i + \sum_{j \in J} a_j t_j = v + u.$$
    This means that $v = \sum_{i \in I'} x_i + \sum_{j \in J'} a_j t_j$ for some $I' \subseteq I$ and $J' \subseteq J$. Since v occurs in v at depth 0 by the first clause of Definition 28, using the second clause of Definition 28 we may conclude that v occurs in t at depth 0.
    Inductive Step: $d > 0$. Since
    $$t = \sum_{i \in I} x_i + \sum_{j \in J} a_j t_j \xrightarrow{s} v + u,$$

and s is non-empty, we have that $s = a_j s'$ and $t_j \xrightarrow{s'} v + u$, for some $j \in J$. By induction, v occurs in t_j at depth $d - 1$, and therefore in $a_j t_j$ at depth d. Using the second clause of Definition 28 we may conclude that v occurs in t at depth d.

- **Proof of statement 2.** Assume that x occurs in t at depth d, $n \in \mathit{lengths}(\sigma(x))$ for some substitution σ, and n is positive. Since x occurs in t at depth d, by statement 1 of the lemma, we have that $t \xrightarrow{s} x + u$ for some sequence of actions s of length d and term u. By Lemma 2, we have that

$$\sigma(t) \xrightarrow{s} \sigma(x + u) = \sigma(x) + \sigma(u).$$

As $n \in \mathit{lengths}(\sigma(x))$ by our assumptions, $\sigma(x) \xrightarrow{s'} v$ for some sequence of actions s' of length n and term v with no outgoing transitions. Since the length of s' is positive, it follows that $\sigma(t) \xrightarrow{ss'} v$ holds, and thus that $d + n \in \mathit{lengths}(\sigma(t))$, which was to be shown. □

Lemma 30. *Let t be a BCCSP term with $\mathit{depth}(t) < m$, and let σ be a closed substantial substitution such that $\mathit{lengths}(\sigma(t)) \subseteq \{n + m, n + 2m\}$, for some $n \geqslant 0$. Then t has a unique depth allocation.*

Proof. Suppose a variable x occurs at depths d_1 and d_2 in t. Let $\mathit{depth}(\sigma(x)) = d$. Since σ is a substantial substitution, d is positive. Then, by Lemma 29(2) and the proviso of Lemma 30, we have that

$$\{d_1 + d, d_2 + d\} \subseteq \mathit{lengths}(\sigma(t)) \subseteq \{n + m, n + 2m\}.$$

(The proof of the first inclusion uses that $d > 0$.) As $|d_1 - d_2| < m$ holds by our assumption that $\mathit{depth}(t) < m$ and Lemma 29(1), this implies $d_1 = d_2$. □

The proof above is the only one where we use that the substitutions are substantial.

Definition 31. For $m, \ell \geqslant 0$, define the operator $\_;_m a^\ell$ on closed BCCSP terms recursively by
- $(\Sigma_{i=1}^k a_i p_i);_{m+1} a^\ell = \Sigma_{i=1}^k a_i (p_i;_m a^\ell)$,
- $(bp + q);_0 a^\ell = bp + q$,
- $\mathbf{0};_0 a^\ell = a^\ell \mathbf{0}$.

Recall that we consider terms modulo associativity and commutativity of +, and modulo absorption of $\mathbf{0}$ summands. Hence any closed BCCSP term with depth 0 can be written as $\mathbf{0}$. Thus, the operator $;_m a^\ell$ adds a sequence of ℓ a-transitions to every state at depth m from which no transitions are possible.

In the remainder of this section, we shall tacitly assume, without loss of generality, that a is the only action occurring in terms. This is justified because the closed terms that we shall use in our proof of Theorem 38 to follow are over action a, and it is easy to see that every closed inequational derivation from an axiom system that is sound with respect to $\preceq_1^T$ proving an inequation $p \sqsubseteq q$, with p and q terms over action a, only uses terms over action a.

Lemma 32. *Let p be a closed BCCSP term, and let $\ell, m, n \geqslant 0$. If $\mathit{depth}(p) < n + m + \ell$ then*

$$p \models (\langle a \rangle \neg)^n \langle a \rangle^m \neg \langle a \rangle \top \Leftrightarrow p;_{n+m} a^\ell \models (\langle a \rangle \neg)^n \langle a \rangle^{m+\ell} \top.$$

Proof. Note, first of all, that the following holds, for each $k \in \mathbb{N}$ and closed BCCSP term q':

$$\exists q\,(p \xrightarrow{a} q \wedge q' = q;_k\, a^\ell) \Leftrightarrow p;_{k+1} a^\ell \xrightarrow{a} q'. \tag{2}$$

We prove the lemma by induction on $n + m$.

- *Case*: $n + m = 0$. Then

$$\begin{aligned} p \models \neg\langle a\rangle\top &\Leftrightarrow p = \mathbf{0} && \text{(as } p \text{ is over action } a) \\ &\Leftrightarrow p;_0\, a^\ell \models \langle a\rangle^\ell\top && \text{(because } depth(p) < \ell). \end{aligned}$$

- *Case*: $n = 0$, $m > 0$. Then

$$\begin{aligned} p \models \langle a\rangle^m\neg\langle a\rangle\top &\Leftrightarrow \exists q\,(p \xrightarrow{a} q \models \langle a\rangle^{m-1}\neg\langle a\rangle\top) \\ &\Leftrightarrow \exists q'\,(p;_m\, a^\ell \xrightarrow{a} q' \models \langle a\rangle^{m+\ell-1}\top) \\ &\Leftrightarrow p;_m\, a^\ell \models \langle a\rangle^{m+\ell}\top, \end{aligned}$$

  where the second equivalence follows by (2) and the inductive hypothesis, using that $q' = q;_{m-1}\, a^\ell$ and $depth(q) < m + \ell - 1$.
- *Case*: $n > 0$. Then,

$$\begin{aligned} p \models (\langle a\rangle\neg)^n\langle a\rangle^m\neg\langle a\rangle\top &\Leftrightarrow \exists q\,(p \xrightarrow{a} q \not\models (\langle a\rangle\neg)^{n-1}\langle a\rangle^m\neg\langle a\rangle\top) \\ &\Leftrightarrow \exists q'\,(p;_{n+m}\, a^\ell \xrightarrow{a} q' \not\models (\langle a\rangle\neg)^{n-1}\langle a\rangle^{m+\ell}\top) \\ &\Leftrightarrow p;_{n+m}\, a^\ell \models (\langle a\rangle\neg)^n\langle a\rangle^{m+\ell}\top, \end{aligned}$$

  where the second equivalence follows by (2) and the inductive hypothesis, using that $q' = q;_{n+m-1}$ a^ℓ and $depth(q) < n + m + \ell - 1$. □

The following example shows that in Lemma 32 the hypothesis $depth(p) < n + m + \ell$ cannot be omitted.

Example 33. If $\ell > 0$, then $a^{m+\ell} \not\models \langle a\rangle^m\neg\langle a\rangle\top$. On the other hand,

$$a^{m+\ell};_m\, a^\ell = a^{m+\ell} \models \langle a\rangle^{m+\ell}\top.$$

Lemma 34. *Let σ be a closed substitution, and let t be a BCCSP term with a unique depth allocation and $depth(t) < k$. Let σ' be a closed substitution with $\sigma'(x) = \sigma(x);_{k-d}\, a^\ell$ whenever x occurs at depth d in t. Then*

$$\sigma'(t) = \sigma(t);_k\, a^\ell.$$

Proof. We apply induction on k.

- *Base Case*: $k = 0$. This base case is vacuous, since there is no term whose depth is smaller than 0.
- *Inductive Step*: $k > 0$. We begin by proving that $\sigma'(v) = \sigma(v);_k\, a^\ell$ for each summand v of t.
  - Consider a summand x of t. Since x occurs at depth 0 in t, the definition of σ' yields that $\sigma'(x) = \sigma(x);_k\, a^\ell$.

- Consider a summand au of t. Since $\sigma'(y) = \sigma(y);_{k-e-1} a^\ell$ for variables y that occur at depth e in u, and $depth(u) < k-1$, by induction we may infer that $\sigma'(u) = \sigma(u);_{k-1} a^\ell$. Hence $\sigma'(au) = a(\sigma(u);_{k-1} a^\ell) = \sigma(au);_k a^\ell$.

Since $\sigma'(v) = \sigma(v);_k a^\ell$ holds for all summands v of t, it follows that $\sigma'(t) = \sigma(t);_k a^\ell$, which was to be shown. □

Remark 35. The assumption that $depth(t)$ be smaller than k in the statement of the above lemma is necessary. Take, for instance, $k = 1$, $t = a + x$, and $\sigma(x) = a^2$. Then, if ℓ is positive,

$$\sigma(t);_1 a^\ell = a^{\ell+1} + a^2 \neq a + a^2 = \sigma'(t).$$

Note that $depth(t) = 1$.

Lemma 36. *Let σ be a closed substitution, and let t be a BCCSP term with a unique depth allocation, $depth(t) < n+m$ and $depth(\sigma(t)) < n+m+\ell$, for some $\ell, m, n \geqslant 0$. Let σ' be a closed substitution with $\sigma'(x) = \sigma(x);_{n+m-d} a^\ell$ whenever x occurs at depth d in t. Then*

$$\sigma(t) \models (\langle a\rangle\neg)^n \langle a\rangle^m \neg\langle a\rangle\top \Leftrightarrow \sigma'(t) \models (\langle a\rangle\neg)^n \langle a\rangle^{m+\ell}\top.$$

Proof. Since $depth(t) < n+m$, Lemma 34 yields that $\sigma'(t) = \sigma(t);_{n+m} a^\ell$. Since $depth(\sigma(t)) < n+m+\ell$, Lemma 36 now follows directly from Lemma 32. □

Note that the formula $(\langle a\rangle\neg)^n \langle a\rangle^{m+\ell}\top$ is contained in the language $\mathcal{L}_{n+1}$ that gives a modal characterization of the equivalence $=^T_{n+1}$. (See Definition 13 and Theorem 14.)

The following lemma will be a key ingredient in the proof of Theorem 38 to follow. As mentioned previously, it will be used to show that, subject to technical conditions, terms related by closed substantial substitution instances of axioms in a finite axiom system that is sound for $(n+1)$-nested trace equivalence, for $n \geqslant 1$, either both satisfy an appropriately chosen formula in the language $\mathcal{L}_{n+2}$ or none of them does.

Lemma 37. *Let t_1, t_2 be a pair of BCCSP terms with $depth(t_i) < m$, for $i = 1, 2$, such that the equation $t_1 \approx t_2$ is sound for $(n+1)$-nested trace equivalence, for some $n \geqslant 0$. Furthermore, let σ be a closed substantial substitution with $lengths(\sigma(t_i)) \subseteq \{n+m, n+2m\}$ for $i = 1, 2$. Then*

$$\sigma(t_1) \models (\langle a\rangle\neg)^n \langle a\rangle^m \neg\langle a\rangle\top \Leftrightarrow \sigma(t_2) \models (\langle a\rangle\neg)^n \langle a\rangle^m \neg\langle a\rangle\top.$$

Proof. Since $lengths(\sigma(t_i)) \subseteq \{n+m, n+2m\}$, for $i = 1, 2$, we have that

$$lengths(\sigma(t_1 + t_2)) \subseteq \{n+m, n+2m\}$$

also holds. Thus, by Lemma 30, the term $t_1 + t_2$ has a unique depth allocation. Let σ' be a closed substitution with $\sigma'(x) = \sigma(x);_{n+m-d} a^{m+1}$ whenever x occurs at depth d in $t_1 + t_2$. Using Lemma 36 (with $\ell = m+1$) for the vertical arrows, and the soundness of $t_1 \approx t_2$ for $=^T_{n+1}$ and the modal characterization of $=^T_{n+1}$ (Theorem 14) for the horizontal one, we obtain

$$\begin{array}{ccc} \sigma(t_1) \models (\langle a\rangle\neg)^n\langle a\rangle^m\neg\langle a\rangle\top & & \sigma(t_2) \models (\langle a\rangle\neg)^n\langle a\rangle^m\neg\langle a\rangle\top \\ \Updownarrow & & \Updownarrow \\ \sigma'(t_1) \models (\langle a\rangle\neg)^n\langle a\rangle^{2m+1}\top & \Leftrightarrow & \sigma'(t_2) \models (\langle a\rangle\neg)^n\langle a\rangle^{2m+1}\top. \end{array}$$

This completes the proof of the lemma. □

After this sequence of preparatory lemmas, we are now ready to prove the promised result to the effect that none of the n-nested simulation and trace equivalences (for $n \geqslant 2$), and none of the n-nested simulation and trace preorders (for $n \geqslant 3$) are finitely based over BCCSP.

Theorem 38. *BCCSP modulo* $=^T_n$ *or* $\leftrightarrows_n$, *for* $n \geqslant 2$, *or* $\preceq^T_n$ *or* $\overset{\subseteq}{\to}_n$, *for* $n \geqslant 3$, *is not finitely (in)equationally axiomatizable.*

Proof. Let E be a finite inequational axiomatization for BCCSP. Pick a positive integer m such that

$$m > \max\{depth(t), depth(u) \mid (t \sqsubseteq u) \in E\}.$$

Let p_n and q_n be defined, for each $n \in \mathbb{N}$, as in Example 6. For ease of reference, we recall that:

$$\begin{array}{ll} p_0 \;\; = a^{2m-1}\mathbf{0} & q_0 \;\; = a^{m-1}\mathbf{0} \\ p_{n+1} = ap_n + aq_n & q_{n+1} = ap_n. \end{array}$$

As argued in Example 6, for every $n \geqslant 1$, we have that $p_n \overset{\subseteq}{\to}_n q_n$, and thus

$$q_n \overset{\subseteq}{\to}_{(n+1)} p_n.$$

Let $\psi_1 = \langle a\rangle^m\neg\langle a\rangle\top$ and $\psi_{n+1} = \langle a\rangle\neg\psi_n$. Note that the formula ψ_n is contained in $\mathcal{L}_{n+1}$, for each $n \geqslant 1$, and that ψ_{n+1} is the formula mentioned in the statement of Lemma 37. By induction on $n \geqslant 1$ one checks that $p_n \models \psi_n$ but $q_n \models \neg\psi_n$.

We now proceed to use the fact that $p_n \models \psi_n$ but $q_n \models \neg\psi_n$ to argue that the inequation $q_n \sqsubseteq p_n$ cannot be proven from any finite set of equations that is sound for $=^T_n$. To this end, suppose that E is sound for $=^T_n$ (which, by Proposition 11, is certainly the case if E is sound for $\leftrightarrows_n$, $\preceq^T_{n+1}$ or $\overset{\subseteq}{\to}_{n+1}$), where $n \geqslant 2$. We show that E is incomplete for $\overset{\subseteq}{\to}_{n+1}$ (and thus certainly for $=^T_n$, $\leftrightarrows_n$, and $\preceq^T_{n+1}$ by Proposition 11), because $E \nvdash q_n \sqsubseteq p_n$. This follows immediately from the following:

Claim 39. *Assume that* $E \vdash p \sqsubseteq q$ *and* $lengths(q) \subseteq \{n+m-1, n+2m-1\}$. *Then*

$$p \models \psi_n \Leftrightarrow q \models \psi_n.$$

In fact, using this claim, we can show that $E \nvdash q_n \sqsubseteq p_n$ as follows. Observe, first of all, that $lengths(p_n)$ is included in $\{n+m-1, n+2m-1\}$, for each $n \in \mathbb{N}$. (In fact, $lengths(p_n)$ equals $\{n+m-1, n+2m-1\}$, for each $n \geqslant 1$.) We have already observed that $p_n \models \psi_n$ but $q_n \models \neg\psi_n$. Thus, by the above claim, the inequation $q_n \sqsubseteq p_n$ cannot be derived from E.

Proof of the claim. We use induction on the length of the derivation of $p \sqsubseteq q$ from E. The cases of reflexivity and transitivity are trivial, using the soundness of E with respect to $=^T_n$ and that, by Lemma 18(a), $p =_n^T q$ implies $lengths(p) = lengths(q)$, for each $n \geqslant 2$. The case that $p \sqsubseteq q$ is a closed substantial substitution instance of an axiom in E has been dealt with by Lemma 37. What remains to consider is closure under contexts: if the claim holds for $p \sqsubseteq q$ it needs to be shown for $p + r \sqsubseteq q + r$, for every closed BCCSP term r over action a, and for $ap \sqsubseteq aq$. The first of these follows trivially by the observation that

$$p + r \models \psi_n \text{ iff } p \models \psi_n \text{ or } r \models \psi_n.$$

For the second, the soundness of E yields $p =_n^T q$. Using the modal characterization of $=_n^T$, and that ψ_{n-1} is contained in $\mathcal{L}_n$, we have that

$$p \models \psi_{n-1} \Leftrightarrow q \models \psi_{n-1}.$$

Since $\psi_n = \langle a \rangle \neg \psi_{n-1}$, it follows that

$$ap \models \psi_n \Leftrightarrow aq \models \psi_n,$$

which was to be shown. □

Remark 40. If E contains the axiom $ax \sqsubseteq ax + a$, which is sound for $\overset{\subseteq}{\rightarrow}_2$, we have that $E \vdash a^{2m} \sqsubseteq a^{m-1}(a^{m+1} + a)$. As $a^{m-1}(a^{m+1} + a) \models \psi_1$ but $a^{2m} \not\models \psi_1$, the proof above, and the claim in particular, does not apply to $\preceq_2^T$ and $\overset{\subseteq}{\rightarrow}_2$.

Indeed, three different proofs appear to be needed to establish all of our non-finite axiomatizability results. In particular, the proofs of non-finite axiomatizability for the possible futures and 2-nested simulation preorders are necessarily distinct, because if the set of actions A is a singleton, then there is a finite axiom system that is sound for the possible futures preorder and complete for the 2-nested simulation preorder. This we now proceed to show.

Assume that a is the only action, and consider the axiom system E_{PF} that contains the equations in Table 2, and the inequation

$$a(x + y) \sqsubseteq ax + ay. \tag{3}$$

It is not too hard to see that E_{PF} is sound for the possible futures preorder. In fact, for all closed BCCSP terms p, q,

- the terms $a(p + q)$ and $ap + aq$ have the same traces, and
- if a is the only action, then $p + q$ has the same set of traces as either p or q.

It follows that Eq. (3) is sound with respect to the possible futures preorder, if a is the only action.

We shall now show that E_{PF} is complete for the 2-nested simulation preorder over the collection of closed BCCSP terms over action a. The following lemma will play a key role in the proof of this result.

Lemma 41. *Let p, q be closed BCCSP terms over action a. Assume that $depth(p) \leqslant depth(q)$. Then*

$$E_{\mathrm{PF}} \vdash q \sqsubseteq q + p.$$

Proof. By induction on the sum of the "sizes" of the closed BCCSP terms p, q. We proceed by a case analysis on the form p may take.

- *Case* $p = \mathbf{0}$. In this case, $E_{\mathrm{PF}} \vdash q \approx q + p$ follows immediately from axiom A4 in Table 2.
- *Case* $p = ap'$, *for some* p'. Assume that $q = \sum_{j\in J} aq_j$, for some finite index set J and closed terms q_j over action a ($j \in J$). Since $depth(p) \leqslant depth(q)$ by our assumptions, there is an index $j \in J$ such that $depth(p') \leqslant depth(q_j)$. By the inductive hypothesis, we have that

  $$E_{\mathrm{PF}} \vdash q_j \sqsubseteq q_j + p'.$$

  Hence,

  $$\begin{aligned} E_{\mathrm{PF}} \vdash aq_j &\sqsubseteq a(q_j + p') \\ &\sqsubseteq aq_j + ap' \qquad \text{(By (3))}. \end{aligned}$$

  The claim now follows using closure with respect to BCCSP contexts.
- *Case* $p = p_1 + p_2$, *for some* p_1, p_2 *different from* $\mathbf{0}$. Since $depth(p) \leqslant depth(q)$ by our assumptions, we have $depth(p_i) \leqslant depth(q)$ for $i = 1, 2$. By the inductive hypothesis, we may infer that

  $$E_{\mathrm{PF}} \vdash q \sqsubseteq q + p_i,$$

  for $i = 1, 2$. Thus,

  $$E_{\mathrm{PF}} \vdash q \sqsubseteq q + p_2 \sqsubseteq q + p_1 + p_2,$$

  which was to be shown. □

We are now ready to prove that the axiom system E_{PF} is complete for the 2-nested simulation preorder over closed BCCSP terms over action a.

Theorem 42. *Let* p, q *be closed BCCSP terms over action* a. *Assume that* $p \stackrel{\subsetneq}{\rightarrow}_2 q$. *Then*

$$E_{\mathrm{PF}} \vdash p \sqsubseteq q.$$

Proof. We prove the claim by induction on the depth of p. Let $p = \sum_{i\in I} ap_i$ and $q = \sum_{j\in J} aq_j$, for some finite index sets I and J and closed terms p_i ($i \in I$) and q_j ($j \in J$) over action a. Note that, as $p \stackrel{\subsetneq}{\rightarrow}_2 q$, the depth of q is equal to that of p (Lemma 18(b)).

Let $i \in I$. Then, since $p \stackrel{\subsetneq}{\rightarrow}_2 q$, there is an index j_i such that $p_i \stackrel{\subsetneq}{\rightarrow}_2 q_{j_i}$ (Proposition 5). Since the depth of p_i is smaller than that of p, by our inductive hypothesis it follows that the inequation $p_i \sqsubseteq q_{j_i}$ can be proven from E_{PF}. Since this holds for each $i \in I$, we have that

$$E_{\mathrm{PF}} \vdash p \sqsubseteq \sum_{i\in I} aq_{j_i}.$$

To conclude the proof, it suffices only to show that

$$E_{\mathrm{PF}} \vdash \sum_{i\in I} aq_{j_i} \sqsubseteq q.$$

To this end, note that, since E_{PF} is sound with respect to the possible futures preorder, and the inequation $p \sqsubseteq \sum_{i\in I} aq_{j_i}$ is derivable from it, the terms p and $\sum_{i\in I} aq_{j_i}$ have the same depth (Lemma 18(b)). As previously observed, p and q also have the same depth. Write now

$$q = \sum_{i \in I} aq_{j_i} + r,$$

where r is the sum of all the summands of q not occurring in $\sum_{i \in I} aq_{j_i}$. By the previous observations, we have that

$$depth(r) \leqslant depth(q) = depth\left(\sum_{i \in I} aq_{j_i}\right).$$

Lemma 41 now yields that

$$E_{\mathrm{PF}} \vdash \sum_{i \in I} aq_{j_i} \sqsubseteq \sum_{i \in I} aq_{j_i} + r = q,$$

completing the proof. □

6. Finitely based approximations of bisimulation equivalence

The results presented in the previous sections show that none of the nested simulation and trace equivalences afford finite equational axiomatizations over the language BCCSP, even in the presence of a singleton action set. The only exceptions to this rule are the 0-nested and 1-nested simulation and trace equivalences, which happen to be the universal relation, simulation and trace equivalence. Interestingly, however, as shown in [17,18], the intersection of all of the n-nested simulation or trace equivalences or preorders over image-finite labelled transition systems, and therefore over the language BCCSP, is bisimulation equivalence. Hennessy and Milner proved in [18] that bisimulation equivalence is axiomatized over the language BCCSP by the equations in Table 2. It follows that this fundamental behavioural equivalence, albeit finitely based over BCCSP, is the limit of sequences of relations that do not afford finite equational axiomatizations themselves. This is by no means the only example from process theory of a "discontinuous" property of a behavioural equivalence—i.e., of a property that "appears at the limit", but is not afforded by its finite approximations. Other examples of this phenomenon may be found in, e.g., the study of decidability properties of behavioural equivalences over classes of infinite state processes. For instance, as shown in [5,9,10], bisimulation equivalence is decidable over the languages BPA and BPP, but none of the other notions of behavioural equivalence in the linear time-branching time spectrum is—see, e.g., the references [16,21].

It is a natural question to ask at this point whether bisimulation equivalence over BCCSP is the limit of some sequence of finitely based behavioural equivalences that have been presented in the literature. We shall now argue that this does hold, provided that the set of actions is finite.

As stated in Section 2.4, the n-nested trace equivalences were introduced in [18, p. 147] as a a tool to define bisimulation equivalence [25,29]. In [18] Hennessy and Milner introduced another sequence of relations that approximate bisimulation equivalence. These were defined thus:

Definition 43. For every $n \geqslant 0$, the relations $=_n^A$ are defined inductively over closed BCCSP terms thus:

- $p =_0^A q$ for every p, q;
- $p =_{n+1}^A q$ iff for every action $a \in A$:

- if $p \xrightarrow{a} p'$ then there is a q' such that $q \xrightarrow{a} q'$ and $p' =_n^A q'$, and
- if $q \xrightarrow{a} q'$ then there is a p' such that $p \xrightarrow{a} p'$ and $p' =_n^A q'$.

Note that, unlike the n-nested trace equivalences $=_n^T$, the relations $=_n^A$ explore the behaviour of BCCSP terms only up to "depth n." As shown by Hennessy and Milner, over image-finite labelled transition systems, bisimulation equivalence is the intersection of all of the relations $=_n^A$. Moreover, each of the $=_n^A$ is preserved by the operators of Milner's CCS, and *a fortiori* by those of BCCSP.

Our order of business will now be to offer a complete axiomatization of the relations $=_n^A$ over closed BCCSP terms. Let Ax denote the axiom system in Table 2. We shall now show how to inductively construct a family of axiom systems E_n, for $n \geqslant 0$, with the following property:

Theorem 44. *Let p, q be closed BCCSP terms. Then $p =_n^A q$ if, and only if, $Ax \cup E_n \vdash p \approx q$.*

The axiom systems E_n, for $n \geqslant 0$, will be finite, if so is the set of actions A. In what follows we assume that the set of variables is $\{x_1, x_2, \ldots\}$.

Definition 45. For each $n \geqslant 0$, we define the axiom system E_n thus:

$$E_0 = \{x_1 \approx x_2\} \quad \text{and}$$
$$E_{n+1} = \{a(t + x_{n+3}) \approx a(u + x_{n+3}) \mid a \in A, (t \approx u) \in E_n\}.$$

Note that, if A is a finite set set containing, say, k actions, then the axiom system E_n contains k^n equations, for each $n \geqslant 0$. Moreover, observe for later use that, for each $n \geqslant 0$, the axioms in E_n only use variables $x_1, \ldots, x_{n+2}$.

We shall now show that Theorem 44 does hold for the previously defined axiom systems E_n. Since the soundness of each of the axioms in E_n can easily be shown by induction on n, using the aforementioned congruence properties of the relations $=_n^A$, we shall limit ourselves to presenting a proof of the completeness of $Ax \cup E_n$ with respect to $=_n^A$ over closed BCCSP terms. The following lemma will be useful in such a proof.

Lemma 46. *Let $n \geqslant 0$, and let p, q be closed BCCSP terms. Assume that $Ax \cup E_n \vdash p \approx q$. Then $Ax \cup E_{n+1} \vdash ap \approx aq$, for each action $a \in A$.*

Proof. Assume that $Ax \cup E_n \vdash p \approx q$, for some closed BCCSP terms p, q. Recall that this means that there is a sequence $p_1 \approx \cdots \approx p_k$ ($k \geqslant 1$) such that

- $p = p_1$,
- $q = p_k$ and
- $p_i = C[\sigma(t)] \approx C[\sigma(u)] = p_{i+1}$ for some closed substitution σ, context $C[]$ and pair of terms t, u with $t \approx u$ or $u \approx t$ an axiom in $Ax \cup E_n$ ($1 \leqslant i < k$).

We prove that $Ax \cup E_{n+1} \vdash ap \approx aq$, for each action $a \in A$, by induction on k.

- *Base Case*: $k = 1$. In this case we have that $p = q$. Thus, the equation $p \approx q$ is provable from Ax, and so is $ap \approx aq$.

- *Inductive Step*: $k > 1$. By the inductive hypothesis, the equation $ap \approx ap_{k-1}$ is provable from the axiom system $Ax \cup E_{n+1}$. Since $ap_k = aq$, to complete the proof, we are therefore left to prove that

$$Ax \cup E_{n+1} \vdash ap_{k-1} \approx ap_k. \tag{4}$$

To this end, recall that
- $p_{k-1} = C[\sigma(t)]$ and
- $p_k = C[\sigma(u)]$,

for some closed substitution σ, context $C[]$ and pair of terms t, u with $t \approx u$ or $u \approx t$ an axiom in $Ax \cup E_n$. In case an axiom from Ax or its symmetric counterpart was used, (4) follows immediately from the rule of closure under BCCSP contexts. The proof for the case when $t \approx u$ is an axiom in E_n proceeds by a case analysis on the form of the context $C[]$.

(a) *Case 1*: Suppose $C[]$ is of the form $C'[b([] + r)]$, for some action b and closed term r. In this case, it is sufficient to show that

$$Ax \cup E_{n+1} \vdash b(\sigma(t) + r) \approx b(\sigma(u) + r)$$

as (4) will then follow by applying the rule of closure under BCCSP contexts repeatedly. To this end, let σ' be the closed substitution that maps variable x_{n+3} to r, and acts like σ on all of the other variables. Using the axioms in $Ax \cup E_{n+1}$, we have that

$$\begin{aligned} b(\sigma(t) + r) &\approx \sigma'(b(t + x_{n+3})) \quad (\text{as } x_{n+3} \notin var(t)) \\ &\approx \sigma'(b(u + x_{n+3})) \quad (\text{as } b(t + x_{n+3}) \approx b(u + x_{n+3}) \in E_{n+1}) \\ &\approx b(\sigma(u) + r) \quad (\text{as } x_{n+3} \notin var(u)), \end{aligned}$$

which was to be shown.

- *Case 2*: Suppose $C[]$ is of the form $[] + r$, for some closed term r. In this case, letting σ' be defined as above, and using the axioms in $Ax \cup E_{n+1}$, we have that

$$\begin{aligned} ap_{k-1} &\approx a(\sigma(t) + r) \\ &\approx \sigma'(a(t + x_{n+3})) \quad (\text{as } x_{n+3} \notin var(t)) \\ &\approx \sigma'(a(u + x_{n+3})) \quad (\text{as } a(t + x_{n+3}) \approx a(u + x_{n+3}) \in E_{n+1}) \\ &\approx a(\sigma(u) + r) \quad (\text{as } x_{n+3} \notin var(u)) \\ &\approx ap_k, \end{aligned}$$

which was to be shown.

The remaining case, viz. when $u \approx t$ an axiom in E_n, is similar. □

We are now ready to establish the completeness of $Ax \cup E_n$ with respect to $=^A_n$ over closed BCCSP terms, for each $n \geqslant 0$.

The proof is by induction on n. The base case is trivial since the equation $x_1 \approx x_2$ can be used to prove every (closed) equation.

For the inductive step, assume that $Ax \cup E_n$ is complete with respect to $=^A_n$ over closed BCCSP terms, and that $p =^A_{n+1} q$ holds for closed terms p, q. We shall now argue that the equation $p \approx q$ can be derived from the axiom system $Ax \cup E_{n+1}$. Let $p = \sum_{i \in I} a_i p_i$ and $q = \sum_{j \in J} b_j q_j$, for some finite index sets I and J and closed terms $a_i p_i$ $(i \in I)$ and $b_j q_j$ $(j \in J)$. Our order of business will now be to show that

$$Ax \cup E_{n+1} \vdash p \approx p + q \approx q.$$

By symmetry, it is sufficient to show that the equation $p + q \approx q$ is derivable from $Ax \cup E_{n+1}$. To this end, let $i \in I$. Then, since $p =^{A}_{n+1} q$, there is an index j_i such that $a_i = b_{j_i}$ and $p_i =^{A}_{n} q_{j_i}$. Since the axiom system $Ax \cup E_n$ is complete with respect to $=^{A}_{n}$ by our inductive hypothesis, it follows that the equation $p_i \approx q_{j_i}$ can be proven from $Ax \cup E_n$. By Lemma 46, the equation $a_i p_i \approx b_{j_i} q_{j_i}$ can be derived from $Ax \cup E_{n+1}$. As this holds for each index $i \in I$, it follows that $p + q \approx q$ is derivable from $Ax \cup E_{n+1}$, which was to be shown.

The proof of Theorem 44 is now complete.

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